Lecture 3
The Revenue Equivalence Theorem

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Set-up

Class $A$ of auction mechanisms with 4 properties:

1. A buyer can make any bid above a “reserve” price announced by the seller;
2. The winner is the highest-bid buyer;
3. Each bidder is alike (anonymity of auction rules);
4. Each bidder follows the same equilibrium bidding strategy

\[ b_i = \beta(v_i), \quad i = 1, \ldots, N, \]

which is strictly increasing in its argument.
Expected profit for each buyer is simply:

$$(\text{reservation value})(\text{probability of winning}) - (\text{expected payment})$$

Suppose that $N-1$ bidders follow the equilibrium bidding strategy $\beta(\cdot)$ whereas bidder 1 bids $b_1 = \beta(x)$. Then $\beta(v_1)$ is an equilibrium bidding function if bidder 1 maximizes his expected profit by choosing $x = v_1$

(Assume private values to be drawn from the cdf $F(\cdot)$ on $[\underline{v}, \bar{v}]$)

Any auction rule must specify how much bidder 1 is going to pay given $(b_1, \ldots, b_N)$, that is,

$$p_1 = p(b_1, b_2, \ldots, b_N) = p(\beta(x), \beta(v_2), \ldots, \beta(v_N)).$$
Given $b_1 = \beta(x)$, the expected payment by bidder 1 is

$$P(x) = E_{v_2,\ldots,v_N} [p(\beta(x), \beta(v_2), \ldots, \beta(v_N))].$$

Since $\beta(\cdot)$ is assumed to be strictly increasing, bidder 1 is the winner if all other valuations are less than $x$, thus the probability of winning is $F^{N-1}(x)$. Bidder 1’s expected payoff writes

$$\Pi_1(v_1, x) = v_1 F^{N-1}(x) - P(x).$$

If $v^*$ has to be the equilibrium bidding function, then bidder 1 must select $x = v_1$, that is, $\Pi_1(v_1, v_1)$ is the maximal payoff for bidder 1. The following FOC must then be satisfied:
\[
\frac{\partial \Pi_1(v_1, x)}{\partial x} = v_1 \frac{dF^{N-1}(x)}{dx} - P'(x) = 0 \quad \text{at} \quad x = v_1,
\]

for all \( v_1 \geq v^* \), where \( v^* \) is the reservation value for which any bidder is indifferent between submitting the bid \( \beta(v^*) \) and not entering the auction. That is, \( v^* \) satisfies

\[
\Pi_i(v^*, v^*) = v^* F^{N-1}(v^*) - P(v^*) = 0.
\]

At \( x = v_1 \) it has to be true that

\[
v_1 \frac{dF^{N-1}(v_1)}{dv_1} = P'(v_1).
\]
This is a rather standard first-order differential equation. Integrating both sides yields

\[ \int_{v^*}^{v_1} P'(x) \, dx = \int_{v^*}^{v_1} u \frac{d F^{N-1}(u)}{du} + c \Leftrightarrow \]

\[ P(v_1) - P(v^*) = v_1 F^{N-1}(v_1) - v^* F^{N-1}(v^*) - \int_{v^*}^{v_1} F^{N-1}(u) du + c \]

\[ v_1 = v^* \Rightarrow c = 0 \text{ and } P(v^*) = v^* F^{N-1}(v^*) \Rightarrow \]

\[ P(v_1) = v_1 F^{N-1}(v_1) - \int_{v^*}^{v_1} F^{N-1}(u) du. \]
Let us consider the seller’s problem. From her viewpoint, both $v_1$ and $P(v_1)$ are random variables. The seller’s expected payment from bidder 1 is then the expected value of $P(v_1)\) 

$$
\bar{p}_1 \equiv E_{v_1}[p_1] = \int_{v^*}^{v} P(v_1)F'(v_1)
$$

$$
= \int_{v^*}^{v} \left[ v_1 F^{N-1}(v_1) - \int_{v^*}^{v_1} F^{N-1}(u)du \right] F'(v_1)
$$

$$
= \int_{v^*}^{v} v_1 F^{N-1}(v_1) F'(v_1) - \int_{v^*}^{v} \left[ \int_{v^*}^{v_1} F^{N-1}(u)du \right] F'(v_1).
$$

By changing the order of integration we get

$$
\int_{v^*}^{v} \left[ \int_{v^*}^{v_1} F^{N-1}(u)du \right] F'(v_1) = \int_{v^*}^{v} \left[ \int_{v^*}^{v} F'(u)du \right] F^{N-1}(v_1)dv_1.
$$
The intuition of interchanging the order of integration

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{i} v(i, j) = \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} h(i, j)
\]
\[ p_1 = \int_{v^*}^{\bar{v}} \left\{ v_1 F^{N-1}(v_1) F'(v_1) + \left[ F(v_1) - 1 \right] F^{N-1}(v_1) \right\} dv_1 \]
\[ = \int_{v^*}^{\bar{v}} \left[ v_1 F'(v_1) + F(v_1) - 1 \right] F^{N-1}(v_1) dv_1. \]

**Main result**

Assume IID values and suppose that bidders are risk-neutral. The common equilibrium bidding strategy of the family $A$ of auction mechanisms yields the seller an expected revenue of

\[ N \int_{v^*}^{\bar{v}} \left[ v F'(v) + F(v) - 1 \right] F^{N-1}(v) dv, \]

where $v^*$ is the reservation value below which it is unprofitable to submit a bid.
This result does not include any reference to the equilibrium bidding function $\beta(x)$. In a first-price sealed-bid auction $\beta(x)$ is easily derived. Suppose the seller publicly announces a reserve price $r$. Then any bidder $i$ with value $v_i$ has an incentive to participate, that is, $v^* = r$. In this mechanism, for a bidder with value $v$

$$P(v) = \Pr \{ \beta(v) \text{ is the highest bid} \} \beta(v)$$

$$= F^{N-1}(v) \beta(v) \Rightarrow$$

$$\beta(v) = v - \frac{1}{F^{N-1}(v)} \int_r^v F^{N-1}(u) du, \text{ if } v \geq r.$$ 

The first-price and the second-price auctions belong to the class of auctions $A$. 
The next step is to show that the class A is optimal (max seller’s expected revenue) given the appropriate choice of the reserve price. For any auction mechanism there is some implied reservation value \( v^* \) below which buyers will choose not to bid.

Therefore there is a probability \( F^N(v^*) \) that all \( N \) buyers decide not to bid. The total expected return for the seller then writes

\[
v_0 F^N(v^*) + N \int_{v^*}^{v} \left[ v F'(v) + F(v) - 1 \right] F^{N-1}(v) dv,
\]

where \( v_0 \) is the seller’s value for the object. The optimal \( v^* \) must satisfy the FOC:

\[
N v_0 F^{N-1}(v^*) F'(v^*) - \left[ N v^* F'(v^*) + F(v^*) - 1 \right] F^{N-1}(v^*) = 0 \Rightarrow v^* = v_0 + \frac{1 - F(v^*)}{F'(v^*)}.
\]
A sufficient condition for a maximum is

\[
\frac{d}{dv^*} \left[ \frac{1 - F(v^*)}{F'(v^*)} \right] < 0.
\]

**Proposition**

Assume IID values and bidders’ risk neutrality. The members of the family A of auctions mechanisms, which maximize the expected gain of the seller, are those for which the reservation value \( v^* \) below which it is not worthwhile bidding satisfies

\[
v^* = v_0 + \frac{1 - F(v^*)}{F'(v^*)}
\]

(1),

independent of the number of bidders!
Main implications:

- both the first- and the second-price auctions are optimal if the reserve is chosen according to (1);
- the optimal reserve price is strictly higher than the seller’s valuation.

To see the underlying intuition of the last point consider a second-price auction with 2 bidders...
The revenue effect of the optimal reserve price: An example

Two bidders; \( F(v) = v \) on \([0,1]\); \( v_0 = 0 \).

According to the last proposition, it is optimal for the seller to adopt an auction mechanism so that only bidders with reservation values exceeding

\[
\nu^* : \nu^* = 1 - \nu^* \iff \nu^* = \frac{1}{2}
\]

find worthwhile bidding. Therefore \( r = 1/2 \). The optimal bidding strategy in this case becomes

\[
\beta(v) = \frac{1}{2}v + \frac{1}{8}v, \forall v \geq 1/2.
\]
The seller’s expected revenue when adopting the optimal reserve price writes

\[ P^I \equiv E[\beta(v_{(1:N)})] = \int_{1/2}^{1} \left( \frac{1}{2}v + \frac{1}{8} \right) 2v \, dv = \frac{5}{12}. \]

If, instead, the seller chooses \( r = 0 \) (that is, she always sells the good) then \( \beta(v) = (1/2)v \), and the expected revenue is \( 1/3 < 5/12 \).
Another look at the Revenue Equivalence Theorem

\[
P^{II}(v) = \Pr(\text{win}) \cdot E[2^{nd} \text{ highest bid } | \beta(v) \text{ highest bid }] \\
= \Pr(\text{win}) \cdot E[2^{nd} \text{ highest value } | v \text{ highest value }] \\
= \Pr(\text{win}) \cdot E[\tilde{Y}_1 | \tilde{Y}_1 < v] \\
= G(v) \cdot E[\tilde{Y}_1 | \tilde{Y}_1 < v] \\
\int_v^y yg(y)dy \\
= G(v) \cdot \frac{0}{G(v)} \\
= \int_v^y yg(y)dy \\
\text{where } \tilde{Y}_{(1:N-1)} \equiv \max\left(\tilde{\nu}_2, \ldots, \tilde{\nu}_N\right) \text{ and } G(\cdot) \equiv F^{N-1}(\cdot).
\[ P^I(v) = \Pr(\text{win}) \cdot \beta^I(v) \]
\[ = G(v) E[Y_1 | Y_1 < v], \]

which is the same as in a second-price auction.
Because the seller’s expected revenue is just the sum of the *ex ante* - prior to knowing their values - expected payments made by the bidders, the expected revenues in the two auctions must be the same. Let’s see this last point in more detail.

The *ex ante* expected payment of a particular bidder in either auction is

\[
E[P^A(\tilde{v})] = \int P^A(v) f(v) dv = \int \left( \int y g(y) dy \right) f(v) dv,
\]

where \( A = \{I, II\} \). Interchanging the order of integration we get

\[
\int \left( \int f(y) dy \right) v g(v) dv = \int v(1 - F(v)) g(v) dv.
\]
The seller’s expected revenue is then

\[ E[R^A] = N \cdot E[P^A(\tilde{v})] = N \int_0^1 v(1 - F(v))g(v)dv. \]

Note that the density of \( Y_{(2:N)} \), the second-highest of \( N \) values

\[ f_{(2:N)}(v) = N(1 - F(v))f_{(1:N-1)}, \]

and since

\[ f_{(1:N-1)}(v) = g(v) \]

we can write

\[ E[R^A] = \int_0^1 vf_{(2:N)}(v)dv = E[\tilde{Y}_{(2:N)}]. \]
The Simple Economics of Optimal Auctions
(An asymmetric) Set-up

A seller faces $N$ individuals interested in acquiring the good on sale, seller’s value is nought.

From the seller’s viewpoint, each buyer $i$ independently draws his value from a distribution function

$$F_i(v_i) \text{ on } [v_i, \overline{v_i}].$$

Main Question

What sales mechanism will maximize the seller’s expected profits?
The solution in 3 steps

1. Draw the “demand function” for each buyer $i$ where

   $v_i$ is on the price axis

   $1 - F_i(v_i) = q$ is on the quantity axis.
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   $1-F_i(v_i) = q$ is on the quantity axis.

2. Calculate and draw the “marginal revenue” curve in the standard fashion, that is,
   
   $$MR_i(v_i) \equiv \frac{d}{dq} \left[ q \cdot F_i^{-1}(1-q) \right] = F_i^{-1}(1-q) - q \cdot \frac{1}{f_i(F_i^{-1}(1-q))}.$$ 

   Rewriting the marginal revenue as function of $v_i$ we get
   
   $$MR_i(v_i) = \begin{cases} v_i - \frac{1-F_i(v_i)}{f_i(v_i)}, & \text{if } v_i \in [v_i, v_i^-] \\ -\infty, & \text{otherwise} \end{cases}$$
3. Consider the following “second marginal revenue auction”:

a) each bidder announces his value (we will prove that there is no incentive to lie);

b) translate each bidder’s value into marginal revenue (assume it to be monotonically increasing);

c) award the object to the bidder with the highest marginal revenue;

d) the price paid is the lowest value this bidder could have announced without losing the auction
Remarks

• If no bidder has a positive MR, then no sale

• If only one bidder, say bidder j, has a positive \( MR_j \), then the selling price \( p \) is equal to \( v^* \), where \( v^* \) solves \( MR_j(v^*) = 0 \)

• If more than one bidder announces a value that translates into positive MR, then the one with the highest MR, say 1, gets the object and pays \( MR^{-1}(MR_{(2)}) \), where \( MR_{(2)} \) is the second-highest MR

• Highest value does not necessarily translate into highest MR (see Myerson (1981))

• “Honest report” is a (weakly) dominant strategy since if a bidder the auction, the amount paid is independent of his report
A Numerical Example

\[ \tilde{\nu}_A \text{ is } U_{[0,10]} \rightarrow q_A \equiv 1 - F_A(\nu_A) = 1 - \frac{1}{10} \nu_A \Rightarrow p_A = 10 - 10q_A. \]

\[ \tilde{\nu}_B \text{ is } U_{[5,20]} \rightarrow q_B \equiv 1 - F_B(\nu_B) = 1 - \frac{\nu_B - 5}{15} \Rightarrow p_B = 20 - 15q_B. \]

\[ MR_A(q_A) = 10 - 20q_A \rightarrow MR_A(\nu_A) = 10 - 20 \left( \frac{10 - \nu_A}{10} \right) = 2\nu_A - 10. \]

\[ MR_B(q_B) = 20 - 30q_B \rightarrow MR_B(\nu_B) = 20 - 30 \left( \frac{20 - \nu_B}{15} \right) = 2\nu_B - 20. \]

\[ MR_A \geq 0 \iff \nu_A \geq 5, \quad MR_B \geq 0 \iff \nu_B \geq 10, \quad \text{ and } MR_A \geq MR_B \iff \nu_A \geq \nu_B - 5. \]

Suppose \( A \) announces \( \nu_A = 8 \) and \( B \) announces \( \nu_B = 12 \).

Then \( MR_A = 6 > 4 = MR_B \). \( A \) wins despite \( \nu_A < \nu_B \) and pays \( p = 7 \) since \( 7 = MR_A^{-1}(MR_B) \).
Price Discriminating Monopolist and Optimal Auctions

Assumptions:

- \( N \) different markets;
- each consumer is interest at most in one unit;
- consumers in market \( i \) have private values as in the first example;
- \( MR_i \) is downward sloping for all \( i = 1, ..., N \).
- marginal cost is zero up to a certain quantity \( \tilde{Q} \) r.v. defined on \( [\underline{Q}, \overline{Q}] \), with density \( h(\cdot) \).

- the monopolist can set prices (and implicitly quantities) contingent on the realization of \( \tilde{Q} \) and seeks to maximize her profits.
Call $p_i(v, Q)$ the probability that consumer $i$ with value $v$ will acquire a unit if the monopolist’s capacity is $Q$

$$
\bar{p}_i(v) \equiv \int_0^Q p_i(v, Q) h(Q) dQ.
$$

The monopolist’s objective function can be formulated as the difference between the expected social value of units sold and the expected consumers’ surplus. The former writes

$$
E[SV] = \int_0^Q h(Q) \sum_{i=1}^N \int_{v_i}^\infty v f_i(v) p_i(v, Q) dv dQ = \sum_{i=1}^N \int_{v_i}^\infty v f_i(v) \bar{p}_i(v) dv.
$$

What about consumers’ surplus? Note that a buyer with the lowest possible value will always get zero surplus since the monopolist will always set a price higher that the lowest possible private value.
Suppose that a buyer’s value is distributed between 0 and 100 and that monopolist charges a price of 50 with probability 0.5 and a price of $100x$ or less, $0.5 < x \leq 1$, with probability $x$, that is,

$$\Pr(\tilde{p} \leq 100x) = x \iff \Pr(\tilde{p} \leq y) = \frac{y}{100}, \quad 50 < y \leq 100.$$ 

*Food for Thought 1 (*)*

Show that this situation might happen if the monopolist sold in a market with demand curve $p = 100 - q$, and the monopolist’s capacity were uniformly distributed between zero and 100.

Thus a buyer with $v = 80$ can expect to buy a unit at $p = 50$ with probability 0.5 and at $50 < p \leq 80$ uniformly distributed with probability 0.3. (***)
For buyer $i$ we can write

$$p_i(v) = \begin{cases} 
0, & \text{if } v < 50 \text{ (lowest possible price)} \\
\frac{1}{100}v, & \text{if } 50 < v \leq 100.
\end{cases}$$

Integrating along the value axis we see that consumer I’s expected surplus can be written as

$$E[CS_i(v)] = \int_{\nu_i}^{\bar{v}} p_i(x) dx.$$ 

The expected surplus for all $N$ buyers writes

$$E[CS] = \int_{Q}^{\bar{Q}} h(Q) \sum_{i=1}^{N} \int_{\nu_i}^{\bar{v}_i} f_i(v) \int_{\nu_i}^{\bar{v}_i} p_i(x,Q) dx dv dQ = \sum_{i=1}^{N} \int_{\nu_i}^{\bar{v}_i} f_i(v) \int_{\nu_i}^{\bar{v}_i} p_i(x) dx dv.$$
Let’s visualize buyer $i$’s expected surplus when his value $v = 80$
The monopolist’s expected revenue is now

\[
E[R] = \sum_{i=1}^{N} \int_{\bar{v}_i}^{v_i} vf_i(v)p_i(v)dv - \sum_{i=1}^{N} \int_{\bar{v}_i}^{v_i} f_i(v)\int_{\bar{x}_i}^{x_i} p_i(x)dxdv.
\]

The monopolist’s problem becomes finally:

\[\text{Max } E[R]\]

\[\text{s.to } \sum_{i=1}^{N} \int_{\bar{v}_i}^{v_i} f_i(v)p_i(v,Q)dv \leq Q, \ \forall Q. \quad (QC)\]

(this is a feasibility constraint on sales relative to capacity)
The decision variables are prices in $N$ markets or, alternatively, the value at which $p_i(v,Q)$ switches from zero to one. Thus there are two additional constraints:

1. $0 \leq p_i(v,Q) \leq 1$;
2. $p_i(\cdot, Q)$ non-decreasing.

Since the monopolist can make the allocation rule contingent on $Q$, we can maximize $E[R]$ for each possible $Q$. Then the problem writes

$$\max E[R(Q)] = \sum_{i=1}^{N} \int_{v_i}^{\bar{v}_i} vf_i(v) p_i(v,Q)dv - \sum_{i=1}^{N} \int_{v_i}^{\bar{v}_i} f_i(v) \int_{v_i}^{\bar{v}_i} p_i(x, Q)dx dv$$

**s.to**

$$(QC)$$
Interchanging the order of integration of the second term in the maximand we get

\[
\sum_{i=1}^{N} \int_{\bar{v}_i}^{\nu_i} p_i(v,Q) \int_{v}^{\nu_i} f_i(x) dx dv = \sum_{i=1}^{N} \int_{v_i}^{\nu_i} [1 - F_i(v)] p_i(v,Q) dv.
\]

The monopolist’s expected revenue becomes

\[
\sum_{i=1}^{N} \int_{\bar{v}_i}^{\nu_i} \left[ \frac{1 - F_i(v)}{f_i(v)} \right] f_i(v) p_i(v,Q) dv = \sum_{i=1}^{N} \int_{v_i}^{\nu_i} MR_i(v) f_i(v) p_i(v,Q) dv.
\]

In order to get a flavor of the problem’s solution, note that \( f_i(v) > 0 \) enters linearly both the objective function and \((QC)\). Moreover, both the objective function and \((QC)\) are linear in \( p_i(v,Q) \).
Since $MR_i$ is increasing in $\nu$, the constraint $\frac{\partial p_i(\nu, Q)}{\partial \nu} \geq 0$ is never binding. The solution of the monopolist’s problem takes the following form.

For each possible value of $Q$, the monopolist should:

1. Sell to all buyers in each market for whom $MR_i(\nu) \geq 0$ if the quantity constraint is not binding.

2. If the quantity constraint is binding, sell units to the buyers with the highest marginal revenues, by choosing prices to equate the marginal revenues of the lowest-valued buyers actually supplied in each market.
Bidders’ risk aversion
$$u : \mathbb{R}_+ \rightarrow \mathbb{R}; \quad u(0) = 0, \quad u'(\cdot) > 0, \quad u''(\cdot) < 0.$$ 

The second-price auction is not affected by risk aversion!

First-price auction: Suppose that the equilibrium bidding strategy is given by an increasing and differentiable function

$$\gamma : [0,1] \rightarrow \mathbb{R}_+, \text{ with } \gamma(0) = 0.$$ 

If all bidders but bidder 1 follow this equilibrium bidding strategy, then bidder 1 will never bid more than $\gamma(1)$.
Given a value \( v \), bidder 1’s problem is to choose \( z \) in \([0,1]\) and \( \gamma(z) \) to

\[
\max_z \left[ G(z)u(v - \gamma(z)) \right].
\]

**FOC:**

\[
g(z) \cdot u(v - \gamma(z)) - G(z) \cdot \gamma'(z) \cdot u'(v - \gamma(z)) = 0.
\]

In a symmetric equilibrium it must be optimal for bidder 1 to choose \( z = v \). Then

\[
\frac{g(v) \cdot u(v - \gamma(v))}{\gamma'(v)} = G(v) \cdot u'(v - \gamma(v)) \Leftrightarrow \\
\gamma'(v) = \frac{u(v - \gamma(v))}{u'(v - \gamma(v))} \cdot \frac{g(v)}{G(v)} \\
\]

(1)
Under risk neutrality, \( u(v) = v \), so that (1) can be rewritten

\[
\beta'(v) = \left[ v - \beta(v) \right] \cdot \frac{g(v)}{G(v)}.
\]

Next notice that if \( u(\cdot) \) is strictly concave and \( u(0)=0 \), then

\[
\frac{u(y)}{u'(y)} > y, \quad \forall y > 0,
\]

that is, average utility > marginal utility. Hence

\[
\gamma'(v) = \frac{u(v - \gamma(v))}{u'(v - \gamma(v))} \cdot \frac{g(v)}{G(v)} > (v - \gamma(v)) \cdot \frac{g(v)}{G(v)}.
\]
The initial conditions $\gamma(0) = \beta(0) = 0$ and the contradiction above imply that

$$\gamma(v) > \beta(v), \quad \forall v > 0.$$
Asymmetries among Bidders
Asymmetric First-Price Auctions with Two Bidders

- Two bidders: 1 and 2
- Private signals: \((X_1, X_2)\) ID according to \((F_1, F_2)\) defined on \([0, \omega_1]\) and \([0, \omega_2]\)
- Suppose there exists an equilibrium in which the 2 bidders follow the strategies \((\beta_1, \beta_2)\) that are strictly increasing and differentiable with inverses

\[
\phi_1 \equiv \beta_1^{-1}, \phi_2 \equiv \beta_2^{-1}
\]

- It must be the case that: \(B_1(0) = 0 = B_2(0)\), and \(B_1(\omega_1) = B_2(\omega_2)\).

Let

\[b^H \equiv \beta_1(\omega_1) = \beta_2(\omega_2)\]
Given that bidder $j = 1,2$ is following the strategy $\mathcal{B}_j$, the expected payoff of bidder $i = 3-j$ when his value is $x_i$ and bids an amount $b < b^H$ is

$$
\Pi_i(b, x_i) = F_j(\phi_j(b))(x_i - b) = H_j(b)(x_i - b)
$$

where $H_j(\cdot) \equiv F_j(\phi(\cdot))$ denotes the distribution of bidder $j$’s bids.

The FOC for bidder $i = 1,2$ requires that for all $b < b^H$

$$
h_j(b)(\phi_i(b) - b) = H_j(b)
$$

for $j = 3-i$ and, as usual,

$$
h_j(b) = H_j'(b) = f_j(\phi_i(b))\phi_j'(b)
$$

is the density of $j$’s bids. The FOC can be rearranged as follows
\[
\phi'_j(b) = \frac{F_j(\phi_j(b))}{f_j(\phi_j(b))} \cdot \frac{1}{\phi_i(b) - b}
\]

A solution to the system of differential equations above - one for each bidder - together with the relevant boundary conditions constitutes an equilibrium of the first-price auction.

Remarks:

• Unfortunately, an explicit solution can be obtained only under particular circumstances;

• We can nonetheless deduce certain properties of the equilibrium bidding function by making some assumptions on the nature of asymmetries.
Weakness Leads to Aggression

Suppose that bidder 1’s values are “stochastically higher” than those of bidder 2. Let us make an even stronger assumption, namely that $F_1$ dominates $F_2$ in terms of the reverse hazard rate, that is

$$\omega_1 \geq \omega_2, \text{ and } \forall x \in (0, \omega_2), \frac{f_1(x)}{F_1(x)} > \frac{f_2(x)}{F_2(x)}.$$

Remark. Reverse hazard rate dominance implies that $F_1$ stochastically dominates $F_2$, that is, $F_1(x) \leq F_2(x)$.

Ex.: $F_1(x) = [F_2(x)]^0$, for some $\theta > 1$.

Under the above assumption, we will call bidder 1 the “strong” bidder and bidder 2 the “weak” bidder.
Proposition

Suppose that the value distribution of bidder 1 dominates that of bidder 2 in terms of reverse hazard rate. Then, in a first-price auction, the “weak” bidder 2 bids more aggressively than the “strong” bidder 1—that is

\[ \beta_1(x) < \beta_2(x), \forall x \in (0, \omega_2). \]

Remark. Since for all \( b, 0 \leq b \leq b^H \), \( \phi_1(b) > \phi_2(b) \) it follows from FOCs that

\[ \frac{H_2(b)}{h_2(b)} = \phi_1(b) - b > \phi_2(b) - b = \frac{H_1(b)}{h_1(b)}, \]

so that the distribution of bidder 1’s bids \( H_1 \) dominates that of bidder 2 \( H_2 \) in terms of the reverse hazard rate.
Bits of Intuition...

- If $B_1(x) > B_2(x)$ $\Rightarrow$ bidder 1’s distribution of bids is stochastically higher than that of bidder 2

- Consider a particular $b$ and suppose that $B_1(x_1) = B_2(x_2) = b$ $\Rightarrow$ $x_1 < x_2$

- That is, the weak bidder faces both a stochastically higher distribution of competing bids and has a higher value

- Since both forces cause bids to be higher, if it were optimal for the strong bidder to bid $b$ when his value is $x_1$, it cannot be optimal for the weak bidder to bid $b$ when his value is $x_2$ $\Rightarrow$ contradiction!

- In other words, the two forces must balance each other: *The weak bidder faces a stochastically higher of competing bids than does the strong bidder, but the value at which any particular bid $b$ is optimal for the weak bidder is lower than it is for the strong bidder.*
First- vs Second-Price Auctions with Asymmetric Bidders: Revenue Ranking and Efficiency

Proposition

TBW